# Solutions to graded problems in Homework 10

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## 4.9.3

As usual, $y(t) = y_0(t) + y_p(t)$.

Homogeneous solution:

The auxiliary polynomial is $r^2 + 9 = 0$, which gives $r = \pm 3i$. Hence:

$$y_0(t) = A\cos(3t) + B\sin(3t)$$

Particular solution:

Guess:

$$y_p(t) = At\cos(3t) + Bt\sin(3t)$$

**Note:** We have to add an extra factor of $t$ because the righ-hand-side of the equation $2\cos(3t)$ coincides with one of the roots $r = 3i$. Hence the name 'resonance term'. Also, if you plugged in the guess $A\cos(3t) + B\sin(3t)$ in the equation, you would get $0$, which cannot equal to $2\cos(3t)$.

If you plug in $y_p$ into the differential equation, you get:

$$\begin{aligned}
y_p'' + 9y_p &= 2\cos(3t) \\
(At\cos(3t) + Bt\sin(3t))'' + 9(At\cos(3t) + Bt\sin(3t)) &= 2\cos(3t) \\
(A\cos(3t) - 3At\sin(3t) + B\sin(3t) + 3Bt\cos(3t))' + 9(At\cos(3t) + Bt\sin(3t)) &= 2\cos(3t) \\
-3A\sin(3t) - 3A\sin(3t) - \cancel{9At\cos(3t)} + 3B\cos(3t) + 3B\cos(3t) \cdots \\
\cdots - \cancel{9Bt\sin(3t)} + \cancel{9At\cos(3t)} + \cancel{9Bt\sin(3t)} &= 2\cos(3t) \\
-6A\sin(3t) + 6B\cos(3t) &= 2\cos(3t) \\
6B\cos(3t) - 6A\sin(3t) &= 2\cos(3t) + 0\sin(3t)
\end{aligned}$$

Equating coefficients, we get $6B = 2, -6A = 0$, which gives: $\boxed{A = 0, B = \frac{1}{3}}$, whence:

$$y_p(t) = \frac{1}{3} t \sin(3t)$$

General solution

Finally:

$$y(t) = y_0(t) + y_p(t) = A\cos(3t) + B\sin(3t) + \frac{1}{3} t \sin(3t)$$

**NOW** plug in the initial conditions $y(0) = 1$, $y'(0) = 0$ (not before) to get: $\boxed{A = 1, B = 0}$, and hence our final answer is:

$$y(t) = \cos(3t) + \frac{1}{3} t \sin(3t)$$

## 6.1.13

Consider the (pre)-Wronskian:

$$\widetilde{W}(x) = \begin{bmatrix} x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{bmatrix}$$

Now choose $x = 1$, and you get:

$$\begin{aligned} W(1) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 12 \\ 0 & 0 & 6 & 24 \end{vmatrix} &= -6 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 2 & 12 \end{vmatrix} + 24 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} \\ &= -6 \left( -2 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 12 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right) + 24 \left( -2 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 6 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right) \\ &= 12 \neq 0 \end{aligned}$$

Since the Wronskian is nonzero at 1, the functions $x, x^2, x^3, x^4$ are linearly independent on $(-\infty, \infty)$.

**Note:** There are two other slick ways of solving this problem:

1) The first one is to 'cancel' out $x$, and just show that $1, x, x^2, x^3$ are linearly independent. Then the Wronskian is much easier to compute.

2) Another way is to consider the bigger set $1, x, x^2, x^3, x^4$. Notice that those polynomials are linearly independent because $1, x, x^2, x^3, \cdots, x^n$ is linearly independent for every $n$, and $x, x^2, x^3, x^4$ is a linearly independent subset of a linearly independent set, hence linearly independent!

## 6.2.9

The auxiliary polynomial is $r^3 - 9r^2 + 27r - 27 = 0$.

Now by the rational roots theorem, the possible zeros of the polynomial are $r = \pm 1, \pm 3, \pm 9, \pm 27$. Notice $r = 3$ works.

Now use long division:

$$\begin{array}{r|l} & X^2 \quad - 6X \quad + 9 \\ \hline X - 3 & X^3 - 9X^2 + 27X - 27 \\ & -X^3 + 3X^2 \\ \hline & -6X^2 + 27X \\ & 6X^2 - 18X \\ \hline & 9X - 27 \\ & -9X + 27 \\ \hline & 0 \end{array}$$

And you get $r^3 - 9r^2 + 27r - 27 = (r-3)(r^2 - 6r + 9) = (r-3)^3 = 0$, which gives $r = 3$ with multiplicity 3. Finally, we get:

$$y(t) = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$$

## 6.2.25

Suppose:

$$c_0 e^{rx} + c_1 x e^{rx} + \cdots + c_{m-1} x^{m-1} e^{rx} = 0$$

Then:

$$e^{rx}\left(c_0 + c_1 x + \cdots + c_{m-1} x^{m-1}\right) = 0$$

Since $e^{rx} \neq 0$, we can cancel it out from the equation, and get:

$$c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} = 0$$

But the functions $1, x, x^2 \cdots, x^{m-1}$ are linearly independent, hence:

$$c_0 = c_1 = \cdots = c_{m-1} = 0$$

TA-DAAA!!! :)